

Decompactification of space or time in large N QCD₂

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Abstract

QCD₂ with fundamental quarks on a cylinder is solved to leading order in the $1/N$ expansion, including the zero mode gluons. As a result of the non-perturbative dynamics of these gauge degrees of freedom, the compact space-time direction gets effectively decompactified. In a thermodynamic interpretation, this implies that there is no pressure of order N and that the chiral condensate of order N is temperature independent. These findings are consistent with confinement of quarks, rule out both chiral and deconfining phase transitions in the finite temperature 't Hooft model, and help to resolve some controversial issues in the literature.

Whereas 2-dimensional field-theoretic models like the Schwinger model [1] or the Gross-Neveu model [2] are by now well understood also at finite temperature [3, 4], the situation is less clear in the case of the 't Hooft model [5], large N QCD₂ with fundamental quarks. In view of the similarity between these models, notably their chiral aspects, and the vast amount of literature on QCD₂, this is rather surprising. Briefly, the present situation is as follows: McLerran and Sen [6] argue that there is no deconfining phase transition, except possibly at infinite temperature. Ming Li [7], using standard finite temperature field theory methods, concludes that chiral symmetry may get restored in the limit $T \rightarrow \infty$. In both of these studies, severe infrared problems were encountered, either in the form of divergent diagrams or ambiguous quark self energies. In a different vein, several studies have addressed QCD₂ on a spatial circle at zero temperature. By interchanging Euclidean time with space and invoking covariance, this can also be reinterpreted as finite temperature calculations for a spatially extended system, even though the techniques used are quite different. Lenz *et al.* [8] observe a chiral phase transition in the massless 't Hooft model at some critical length, strongly reminiscent of the Gross-Neveu model. Dhar *et al.* [9] treat the zero mode gluons in a more ambitious way than ref. [8], using technology from matrix models and string theory, but are not able to fully solve the resulting complicated equations. They propose that the gauge variables get decompactified by the fermions in complete analogy with the Schwinger model [10], a claim which has recently been disputed by Engelhardt [11].

In view of this unsatisfactory state of the art, we have reconsidered the 't Hooft model on a cylinder and, in particular, tried to clarify the role of the zero mode gluons. We shall present here a novel treatment of the leading order in the large N expansion, which seems to resolve the above sketched discrepancies. This opens the way to a controlled study of the $1/N$ corrections expected to reveal the true, “hadronic” physics of the model.

We work canonically on a spatial circle of length L in the gauge $\partial_1 A_1 = 0$, $(A_1)_{ij} = \delta_{ij} \frac{\varphi_i}{gL}$ diagonal in color. The Hamiltonian reads (cf. [8, 12, 13])

$$H = H_g + H_f + H_C , \tag{1}$$

with the gauge field kinetic energy

$$H_g = -\frac{g^2 L}{4} \sum_i \frac{\partial^2}{\partial \varphi_i^2} , \tag{2}$$

the quark kinetic energy

$$H_f = \sum_{n,i} \frac{2\pi}{L} \left(n + \frac{\varphi_i}{2\pi} \right) \left(a_i^\dagger(n) a_i(n) - b_i^\dagger(n) b_i(n) \right) + m \sum_{n,i} \left(a_i^\dagger(n) b_i(n) + b_i^\dagger(n) a_i(n) \right) , \quad (3)$$

and the Coulomb interaction

$$H_C = \frac{g^2 L}{16\pi^2} \sum_{n,i,j} \frac{j_{ij}(n) j_{ji}(-n)}{\left(n - \frac{\varphi_j - \varphi_i}{2\pi} \right)^2} . \quad (4)$$

Here, the $a_i(n), b_i(n)$ denote second quantized right- and left-handed quarks, respectively, and the currents $j_{ij}(n)$ can be taken in the $U(N)$ form at large N ,

$$j_{ij}(n) = \sum_{n'} \left(a_j^\dagger(n') a_i(n' + n) + b_j^\dagger(n') b_i(n' + n) \right) . \quad (5)$$

As explained elsewhere [12, 13], due to the curved configuration space of the φ_i and the $SU(N)$ Haar measure originally appearing in H_g , this Hamiltonian has to be supplemented by the following boundary condition for the wavefunctionals,

$$\Psi(\varphi_1, \dots, \varphi_N; \text{fermions}) = 0 \quad \text{if} \quad \varphi_i = \varphi_j \bmod 2\pi . \quad (6)$$

In ref. [8], quantization was performed after complete classical gauge fixing. In this way, the fact that the φ_i are curvilinear coordinates is missed. This led to the assumption that all the φ_i are frozen at the value π in the large N limit. In the resulting purely fermionic theory, the only remnant of the gluons are antiperiodic boundary conditions for the quarks in the compact space direction. In the meantime, this whole approach has been put on a more rigorous basis by first quantizing in the Weyl gauge and then resolving the Gauss law quantum mechanically [12]. This made it clear that the φ_i are parameters on the group manifold with corresponding Jacobian, the $SU(N)$ reduced Haar measure. When solving the theory, it is then possible to restrict oneself to the smallest region in field space bounded by zeros of the Jacobian, cf. ref. [13] where the consequences for $SU(2)$, $SU(3)$ were explored in the case of adjoint fermions. How can this be generalized to the large N limit? A definite choice of “fundamental domain” obviously means that the φ_i always remain ordered, say $0 \leq \varphi_1 \leq \varphi_2 \leq \dots \leq \varphi_N \leq 2\pi$. If we think of the gluons as particles on a circle, they cannot cross each other and become closely packed in the limit $N \rightarrow \infty$. Their fluctuations are suppressed by $1/N$, simply due to lack of space. The only degree of freedom left, the collective rotation of this “pearl necklace”, is a $U(1)$

factor which anyway is not present in the $SU(N)$ theory. This suggests that the correct choice for the gluon background field as seen by the fermions is not $\varphi_i = \pi$, but rather the continuum limit of the lattice points

$$\varphi_i = 2\pi \frac{i}{N} , \quad i = 1 \dots N . \quad (7)$$

Instead of antiperiodic boundary conditions, the fermions then acquire color dependent boundary conditions which interpolate smoothly between the phases 0 and 2π ,

$$\psi_k(L) = e^{i2\pi k/N} \psi_k(0) , \quad (k = 1, \dots, N) . \quad (8)$$

Note that in such a completely gauge fixed formulation, there is nothing wrong with having a color dependence of this type.

In the thermodynamic limit $L \rightarrow \infty$, both of these choices of the gluon field configuration, $\varphi_i = \pi$ or eq. (7), will become indistinguishable and yield the well-known results. We now show that at finite L , the effects of the gluons on the quarks is radically different in these two cases, and that it is the spread out distribution (7) which is in fact the correct one.

The fermions can be treated in a relativistic Hartree-Fock approximation along the lines explained in ref. [8]. In this approach, all the information about the vacuum is encoded in the Bogoliubov angle $\theta_i(n)$, related to self-consistent Hartree-Fock spinors via

$$u_i(n) = \begin{pmatrix} \cos \theta_i(n)/2 \\ \sin \theta_i(n)/2 \end{pmatrix} , \quad v_i(n) = \begin{pmatrix} -\sin \theta_i(n)/2 \\ \cos \theta_i(n)/2 \end{pmatrix} . \quad (9)$$

The Bogoliubov angles in turn are determined by the gap equation,

$$\frac{2\pi}{L}(n + \alpha_i) \sin \theta_i(n) - m \cos \theta_i(n) + \frac{g^2 L}{16\pi^2} \sum_{n',j} \frac{\sin(\theta_i(n) - \theta_j(n - n'))}{(n' - \alpha_j + \alpha_i)^2} = 0 . \quad (10)$$

Here, we have switched to the slightly more convenient variable $\alpha_i = \frac{\varphi_i}{2\pi} \in [0, 1]$ for the gluons. If $\alpha_i = 1/2$ as chosen in ref. [8], $\theta_i(n)$ becomes i -independent and we recover the old gap equation considered in that work. Now, we assume $\alpha_i = i/N$ and perform the large N limit before solving the gap equation. Since α_i becomes a continuous variable, we replace $\theta_i(n) \rightarrow \theta_\alpha(n)$ and $\sum_j \rightarrow N \int_0^1 d\alpha'$, with the result

$$\frac{2\pi}{L}(n + \alpha) \sin \theta_\alpha(n) - m \cos \theta_\alpha(n) + \frac{Ng^2 L}{16\pi^2} \sum_{n'} \int_0^1 d\alpha' \frac{\sin(\theta_\alpha(n) - \theta_{\alpha'}(n - n'))}{(n' - \alpha' + \alpha)^2} = 0 . \quad (11)$$

This infinite set of coupled integral equations collapses into a single, one-dimensional integral equation, if we set

$$\theta_\alpha(n) = \theta(n + \alpha) . \quad (12)$$

Since n is integer and $\alpha \in [0, 1]$, this step in effect decompactifies the original spatial circle. With this ansatz, the notation $n + \alpha = \nu$, $n - n' + \alpha' = \nu'$ (where ν, ν' are dimensionless, continuous variables) and the substitution $\sum_{n'} \int_0^1 d\alpha' \rightarrow \int_{-\infty}^{\infty} d\nu'$, we obtain

$$\frac{2\pi}{L} \nu \sin \theta(\nu) - m \cos \theta(\nu) + \frac{Ng^2 L}{16\pi^2} \oint d\nu' \frac{\sin(\theta(\nu) - \theta(\nu'))}{(\nu - \nu')^2} = 0 . \quad (13)$$

Guided by what is known from the 't Hooft model in the limit $L \rightarrow \infty$, we have defined the integral as principal value integral. After rescaling the variables via

$$\frac{2\pi}{L} \nu := p , \quad \frac{2\pi}{L} \nu' := p' , \quad (14)$$

where p, p' have the dimension of momenta, we recover exactly the continuum version of the Hartree Fock equation, namely

$$p \sin \theta \left(\frac{Lp}{2\pi} \right) - m \cos \theta \left(\frac{Lp}{2\pi} \right) + \frac{Ng^2}{4} \oint \frac{dp'}{2\pi} \frac{\sin \left(\theta \left(\frac{Lp}{2\pi} \right) - \theta \left(\frac{Lp'}{2\pi} \right) \right)}{(p - p')^2} = 0 . \quad (15)$$

Denoting the Bogoliubov angle of the continuum 't Hooft model by $\theta_{\text{cont}}(p)$, we conclude that

$$\theta(\nu) = \theta_{\text{cont}} \left(\frac{2\pi}{L} \nu \right) , \quad (16)$$

or, in terms of the original, color dependent Bogoliubov angle,

$$\theta_i(n) \approx \theta_{\text{cont}} \left(\frac{2\pi}{L} \left(n + \frac{i}{N} \right) \right) , \quad (N \rightarrow \infty) . \quad (17)$$

This last relation becomes exact in the limit $N \rightarrow \infty$ only. In this limit, the color- and L -dependent Bogoliubov angles for the 't Hooft model on the circle of length L are all given by one universal function, namely the momentum dependent Bogoliubov angle of the 't Hooft model on the infinite line. We emphasize that this universality only holds for the “pearl necklace” type distribution of gauge variables, eq. (7). If the φ_i are all set equal to π , there is no analytic way known how to relate $\theta(n)$ for different L values, but one has to solve the gap equation numerically for each L [8].

The upshot of this simple exercise is the following: In the large N limit, the gluon variables influence the fermion boundary conditions in such a way that the circle gets

replaced by a line; they decompactify space-time. The length L of the spatial circle becomes an irrelevant parameter. To confirm this last point, let us evaluate those ground state expectation values which are of interest for the bulk thermodynamic properties of the system, if one interchanges L and $\beta = 1/T$. This can be done most conveniently with the help of the key relations

$$\begin{aligned}\langle a_i^\dagger(n) a_i(n') \rangle &= \frac{1}{2} \delta_{n,n'} (1 - \cos \theta_i(n)) , \\ \langle b_i^\dagger(n) b_i(n') \rangle &= \frac{1}{2} \delta_{n,n'} (1 + \cos \theta_i(n)) , \\ \langle a_i^\dagger(n) b_i(n') \rangle &= \langle b_i^\dagger(n) a_i(n') \rangle = -\frac{1}{2} \delta_{n,n'} \sin \theta_i(n) .\end{aligned}\tag{18}$$

The vacuum energy density is given by

$$\begin{aligned}\mathcal{E}_{\text{vac}} &= -\frac{1}{L} \sum_{n,i} \left(\frac{2\pi}{L} (n + \alpha_i) \cos \theta_i(n) + m \sin \theta_i(n) \right) \\ &\quad + \frac{g^2}{32\pi^2} \sum_{n,n',i,j} \frac{1 - \cos(\theta_i(n) - \theta_j(n - n'))}{(n' - \alpha_j + \alpha_i)^2} .\end{aligned}\tag{19}$$

It corresponds to the negative of the pressure, in the other picture. Using the same substitutions as above, \mathcal{E}_{vac} can be converted into the standard continuum expression,

$$\begin{aligned}\mathcal{E}_{\text{vac}} &= -\frac{N}{L} \sum_n \int_0^1 d\alpha \left(\frac{2\pi}{L} (n + \alpha) \cos \theta(n + \alpha) + m \sin \theta(n + \alpha) \right) \\ &\quad + \frac{N^2 g^2}{32\pi^2} \int_0^1 d\alpha \int_0^1 d\alpha' \sum_{n,n'} \frac{1 - \cos(\theta(n + \alpha) - \theta(n - n' + \alpha'))}{(n' - \alpha' + \alpha)^2} \\ &= -N \int \frac{dp}{2\pi} (p \cos \theta_{\text{cont}}(p) + m \sin \theta_{\text{cont}}(p)) \\ &\quad + \frac{N^2 g^2}{8} \int \frac{dp}{2\pi} \int \frac{dp'}{2\pi} \frac{1 - \cos(\theta_{\text{cont}}(p) - \theta_{\text{cont}}(p'))}{(p - p')^2} .\end{aligned}\tag{20}$$

Since the right hand side is L -independent, the pressure will be equal to the vacuum pressure at all temperatures, to leading order in $1/N$. Hence, there is no observable pressure of order N , as one would expect from a confined system of quarks. Treating the quark condensate along the same lines, we find

$$\begin{aligned}\langle \bar{\psi} \psi \rangle &= -\frac{1}{L} \sum_{n,i} \sin \theta_i(n) \\ &= -\frac{N}{L} \sum_n \int_0^1 d\alpha \sin \theta(n + \alpha) \\ &= -N \int \frac{dp}{2\pi} \sin \theta_{\text{cont}}(p) .\end{aligned}\tag{21}$$

Once again, the sum over the discrete momenta and the color sum in the large N limit conspire to produce the continuum result, independently of the starting L value. In the alternative thermodynamic view, the condensate does not depend on temperature to leading order in $1/N$. This leaves no room for a chiral phase transition, not even in the limit $T \rightarrow \infty$. Finally, consider the expectation value of the Polyakov loop. Due to our classical treatment and the assumed field configuration, we trivially get zero,

$$\langle \mathcal{P} \rangle = \frac{1}{N} \sum_i e^{i\varphi_i} \rightarrow \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{i\varphi} = 0 . \quad (22)$$

After interchanging space and time, this signals confinement of static charges at any temperature [14]. Physically, one would expect screening by dynamical quarks, but this cannot be seen yet in leading order in the $1/N$ expansion. In the Wilson loop calculation for instance, screening by dynamical quarks involves at least one additional fermion loop. As is well known, such diagrams are suppressed as compared to planar gluonic diagrams by a factor $1/N$. A similar argument holds for the Polyakov loop.

Notice that so far, we have discussed the influence of the gauge fields on the quarks which is indeed dramatic. Vice versa, we do not expect the quarks to influence significantly the gluon zero point motion, again due to the effects of the Jacobian. The kinetic energy H_g will give the same result as in pure Yang Mills theory. Since this contribution to the energy density is L -independent, it again yields zero pressure, reflecting the absence of physical gluonic excitations in 1+1 dimensions.

We now comment on the various studies of the finite temperature 't Hooft model mentioned in the beginning. The mechanism which we propose here has some similarity with the color singlet projection in the partition function as discussed by McLerran and Sen [6], in particular concerning the N -dependence of the thermodynamic potential. In Ming Li's calculation [7], the problem seems to be the quark self energy. In Ref. [8], it was found that the principal value regulated self energy should be supplemented by an additive constant $Ng^2L/48$ which diverges in the limit $L \rightarrow \infty$. This constant drops out of the calculation of color singlet mesons, but cannot be simply ignored at the Hartree Fock level (the principal value prescription makes sense for a first order pole, but not for a second order pole). If we included this infinite constant into Ming Li's calculation (or, equivalently, a finite temperature Hartree Fock calculation), all the thermal factors would trivially vanish and we would also get zero pressure and a T -independent condensate.

In this sense, the finite L calculation presented above and the conventional thermodynamic calculation are fully consistent. Similar arguments have already been put forward heuristically in ref. [6].

As discussed above, the problem with the chiral phase transition seen in ref. [8] is the neglect of Jacobian and consequently freezing of gluons at the point $\varphi_i = \pi$. The finite L version of the theory solved there is apparently not the gauge theory QCD₂, but has to be interpreted as some other interacting fermion theory of Gross-Neveu type with a Coulomb potential instead of a contact interaction. It cannot do full justice to the confinement of quarks. Finally, in [9], it has tacitly been assumed that the Jacobian can be accounted for by treating gluons as non-relativistic fermions. This is certainly true for pure Yang Mills theory where the idea originally came up [15]. In the presence of quarks, the Hamiltonian is not invariant under permutations of the φ_i . As can be seen from eqs. (1–4), it is only invariant if one simultaneously permutes the quark colors — this is a residual gauge transformation. Antisymmetrization of the gluon variables alone is thus not compatible with the time evolution of the system and cannot be used to satisfy the boundary condition (6). The other point in which we differ from ref. [9] is the issue of decompactification: In the Schwinger model on the circle, the pure gauge field is a particle on a circle. This circle gets decompactified by the fermions — the axial charge provides the missing integer part of the coordinate [10]. For reasons discussed in [11], such a phenomenon is hard to conceive in the SU(N) case, unlike what has been put forward in [9]. Interestingly, we have found exactly the opposite effect: The gluons decompactify the spatial circle on which the fermions live, just because they are so much constrained by their own, compact configuration space. This seems to be the mechanism by which a confined system of quarks in 1+1 dimensions avoids wrong thermodynamic behaviour to leading order in $1/N$.

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